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# DYNAMICS OF NON-NEWTONIAN FLUID INTERFACES IN A POROUS MEDIUM: INCOMPRESSIBLE FLUIDS

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## SUMMARY

This paper concerns the applications of frontal advance theory to the dynamics of a moving flat interface in a porous medium, when both displacing and displaced fluids are of power law behaviour. The rheological effects of non-Newtonian behaviour of these fluids on the interface position and its velocity are numerically illustrated and discussed with regard to the practical implications in oil displacement mechanisms. The results obtained should be useful in finding an optimal policy of injection in order to control the dynamics of the moving interface in field projects of enhanced oil recovery floods.

KEY WORDS Non-Newtonian fluid interfaces Porous media Frontal advance theory Power law behaviour Oil displacement mechanisms

## 1. INTRODUCTION

A knowledge of the rheological effects of non-Newtonian bevahiour of the displacing fluid in oil displacement mechanics is of considerable interest in oil reservoir engineering; a high-viscosity fluid—for example, water with dissolved polymer—is injected into the oil reservoir to modify the mobility ratio between the displacing fluid and the displaced one.

As the experimental studies have shown, the injection into the oil reservoir of certain non-Newtonian displacing fluids having a pseudo-plastic behaviour may improve the volumetric sweep efficiency by minimizing the instability effects on the moving interface separating the displaced and displacing fluids. The use of these displacing fluids to control the mobility of the injected water has increased continuously over the last few years, and a recent review reported that a great number of field tests were estimated to be successful from both a technical and an economical point of view.[1] These non-Newtonian displacing fluids, such as polymer solutions and emulsions of oil in water, exhibit in a certain range of shear rate variation a pseudo-plastic behaviour in which the apparent viscosity decreases with increasing shear rate. One obvious consequence of this rheological behaviour of the displacing fluid is a possible new approach for eliminating the viscous fingering effect in oil displacement mechanics, which appears to be directly responsible for the ultimate low oil recovery in water flooding projects. From this point of view, the recent increasing interest in knowing the rheological effects in the flows of non-Newtonian fluids through porous media, in particular on the dynamics of the moving interface, is well justified.[1,2]

A probelm of special interest at the present time in oil reservoir engineering is the determination of the interface position at a given time. Two situations may arise in practice: the fluid injection into the oil reservoir can be carried out at a constant pressure or at a constant flow rate. For the

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case of constant flow rate the interface location is determined by assuming that the two fluids are incompressible; however, the case of constant pressure of injection will require the solution of a system of non-linear equations, as will be shown further on. As shown recently in the literature, the stability conditions for a non-Newtonian fluid interface under constant flow rate of injection lead to a critical interface velocity; whenever the interface velocity is greater than the critical velocity, an unstable interface will occur and hence a viscous fingering effect will arise.[3-6] Obviously, under conditions of constant pressure of injection and production, the interface velocity is unknown. As a result, the dynamics of the moving interface under constant pressure of injection is relevant to the determination of the interface velocity, expressed in terms of its position, and therefore to the interface stability problem as well.[7-10] The objective of this paper is to investigate the dynamics of the moving interface separating non-Newtonian displacing and displaced incompressible fluids of power law behaviour. Sections 2 and 3 deal with the one-dimensional case, while Sections 4 and 5 are devoted to the case of plane radial flow.

## 2. ONE-DIMENSIONAL FLOW

The sketch of the dynamics of a flat moving interface in a linear displacement mechanism is shown in Figure 1.

Under conditions of constant pressure of injection and production the following external boundary conditions arise:

$$x=0, \qquad p_1(0)=p_e=\text{constant}, \tag{1}$$

$$x=L, \qquad p_2(L)=p_w=\text{constant}, \quad p_e>p_w, \tag{2}$$

while at the moving interface, neglecting the surface tension effect, one has

$$x=\xi(t), \qquad p_1(\xi)=p_2(\xi)=p^*, \tag{3}$$

$$v_1(\xi)=v_2(\xi), \tag{4}$$

in which $\xi(t)$ is the interface location and $p^*$ is the pressure at the interface.

As whown previously, the modifigd Darcy's law, including the rheological effects of power law fluids, is expressed as[3]

$$-\frac{\partial p_1}{\partial x}=\frac{\mu_{ef1}}{k_1}v_1^{n_1} \qquad 0<x<\xi(t), \tag{5}$$

$$-\frac{\partial p_2}{\partial x}=\frac{\mu_{ef2}}{k_2}v_2^{n_2}, \quad \xi(t)<x<L, \tag{6}$$

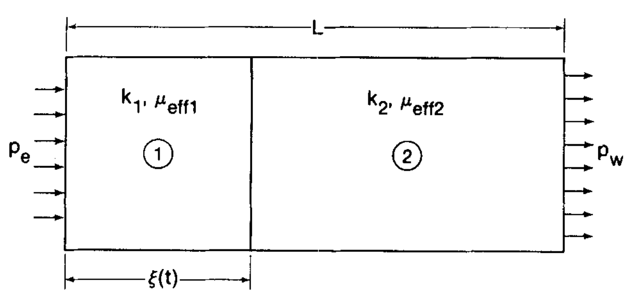


Figure 1. Illustration of flow system in linear displacement mechanism

where

$$\frac{k_i}{\mu_{efi}}=\frac{1}{2H_i}\left(\frac{n_i\varphi}{1+3n_i}\right)^{n_i}\left(\frac{8k_i}{\varphi}\right)^{(n_i+1)/2}, \quad i=1, 2. \tag{7}$$

The notations used are given in Appendix I.

From the continuity equation for incompressible fluids one has $\partial v_i/\partial x=0$, $i=1, 2$, whereas from relations (5) and (6) the pressure distributions ahead of and behind the moving interface may be written, taking into account the conditions specified in (1)–(4), as follows:

$$p_1(x, t)=p_e-(p_e-p^*)\frac{x}{\xi(t)}, \quad 0<x<\xi(t), \tag{8}$$

$$p_2(x, t)=p^*-(p^*-p_w)\frac{x-\xi(t)}{L-\xi(t)}, \quad \xi(t)<x<L. \tag{9}$$

Since the external boundary conditions do not change in time and the two fluids are incompressible, one expects to have a steady flow, i.e. the pressure distributions are time-independent. However, the presence of a moving interface in our flow system shows that the pressure distributions (8) and (9) depend on the interface position $\xi(t)$, which is time-dependent.

From (5), (6), (8) and (9) we have

$$v_1=\left(\frac{k_1}{\mu_{ef1}}\frac{p_e-p^*}{\xi(t)}\right)^{1/n_1} \tag{10}$$

and

$$v_2=\left(\frac{k_2}{\mu_{ef2}}\frac{p^*-p_w}{L-\xi(t)}\right)^{1/n_2}. \tag{11}$$

When these relations are introduced into (4), one obtains

$$\left(\frac{k_1}{\mu_{ef1}}\frac{p_e-p^*}{\xi}\right)^{1/n_2}=\left(\frac{k_2}{\mu_{ef2}}\frac{p^*-p_w}{L-\xi}\right)^{1/n_2}. \tag{12}$$

Relation (12) determines the pressure $p^*$ at the interface location $\xi(t)$.

On the other hand, the interface velocity is determined by the relation

$$V=\varphi\frac{\mathrm{d}\xi}{\mathrm{d}t}=\left(\frac{k_1}{\mu_{ef1}}\frac{\partial p_1}{\partial x}\right)^{1/n_1}_{x=\xi}, \tag{13}$$

so that from the previous relations we have

$$\varphi\frac{\mathrm{d}\xi}{\mathrm{d}t}=\left(\frac{k_1}{\mu_{ef1}}\frac{p_e-p^*}{\xi(t)}\right)^{1/n_1}. \tag{14}$$

Therefore the coupled equations (12) and (14) determine the dynamics of a non-Newtonian fluid flat interface, i.e. the functions $V(t)$ and $\xi(t)$. Integration of this system of non-linear equations requires a numerical procedure, since difficulties associated with finding the exact analytical solution for system (12) and (14) exist in almost all non-Newtonian flows, even for simple geometrical flow systems.

Befeore giving a numerical solution for the general case, we will consider the particular case when $n_1=n_2=n$. This case allows us not only to develop a tractable analytical solution for illustrating the rheological effects on the dynamics of the moving interface, but also to check the results obtained by our numerical procedure, which will be shown in the next section.

When $n_1=n_2=n$, from (12) one obtains

$$p^*=\frac{p_w\xi M+(L-\xi)p_e}{L+(M-1)\xi}, \tag{15}$$

in which

$$M=\frac{\mu_{ef1}k_2}{\mu_{ef2}k_1} \tag{16}$$

is the mobility ratio.

By means of (15), equation (14) becomes

$$\varphi\frac{d\xi}{dt}=\left(\frac{k_1M\Delta p}{\mu_{ef1}[L+(M-1)\xi]}\right)^{1/n}, \qquad \Delta p=p_e-p_w. \tag{17}$$

Assuming that the injection process is started at $t=0$, then at $t=0$ one has $\xi(0)=0$. As a result, integration of (17) leads to

$$T=\frac{n}{(1+n)(M-1)}\left[\left(1+(M-1)\frac{\xi}{L}\right)^{(1+n)/n}-1\right], \tag{18}$$

where $T$ is the dimensionless time, expresssed as

$$T=\left(\frac{k_1M\Delta p}{\mu_{ef1}}\right)^{1/n}\frac{t}{\varphi L^{(1+n)/n}}. \tag{19}$$

From the previous relations, the case of Newtonian displacing and displaced fluids, i.e. $n=1$, yields a well known result, termed in the literature the frontal advance theory:

$$\frac{k_1M\Delta pt}{\mu_1\varphi L^2}=\frac{\xi}{L}+\frac{M-1}{2}\left(\frac{\xi}{L}\right)^2. \tag{20}$$

The interface velocity expressed in terms of its position is obtained from the relation

$$V=\left(\frac{k_1M\Delta p}{\mu_{ef1}[L+(M-1)\xi(t)]}\right)^{1/n}, \tag{21}$$

in which $\xi(t)$ is determined from (18) and $M$ from (16).

A problem of special interest in oil reservoir engineering is the prediction of the time required for the interface to reach a given length of the flow system. This travel time may be determined from (18) when $\xi=L$, $L$ being the distance between injector and producer. It is interesting to remark from (21) that the interface movement is accelerated for $M<1$, while for $M>1$ it is decelerated. When $M=1$ the interface movement will be at a constant velocity, as can be readily seen from (21).

## 3. NUMERICAL SOLUTION

As pointed out earlier, for the general case the integration of the system of coupled non-linear equations (12) and (14) must be carried out numerically. Using the notation

$$Z=\frac{k_1}{\mu_{ef1}}\frac{p_e-p^*}{\xi}, \tag{22}$$

equations (12) and (14) may be written as

$$Z^m+\frac{M\xi}{L-\xi}Z-\frac{k_2\Delta p}{\mu_{ef2}(L-\xi)}=0, \tag{23}$$

$$\frac{d\xi}{dt}=\frac{1}{\varphi}Z^{1/n_1}, \tag{24}$$

with the initial condition $\xi(0)=0$ and the notations

$$\Delta p=p_e-p_w, \qquad m=n_2/n_1. \tag{25}$$

From (22) the pressure at the interface is

$$p^*=p_e-\frac{\mu_{ef1}}{k_1}\xi Z. \tag{26}$$

To start our numerical procedure of solving the system of equations (23) and (24), we take into account the fact that for a short time the interface location is close to its initial position, corresponding to $t=0$, and the pressure $p^*$ at the interface location $x=\xi$ does not differ significantly from the injection pressure $p_e$. In this case, since $p^*\simeq p_e$ in (26), $\xi Z\to 0$, so that (23) yields

$$Z=\left(\frac{k_2\Delta p}{\mu_{ef2}(L-\xi)}\right)^{1/m} \tag{27}$$

and (24) gives us for the interface position at short time the approximate analytical solution

$$\xi(t)=L-L\left[1+\frac{1+n_2}{n_2}\left(\frac{k_2\Delta p}{\mu_{ef2}}\right)^{1/n_2}\frac{t}{L^{(1+n_2)/n_2}}\right]^{n_2/(1+n_2)}. \tag{28}$$

In solving (23) and (24), we look for the positive root of (23). To locate this root, we use the notation

$$F(Z)=Z^m+\frac{M\xi}{L-\xi}Z-\frac{k_2\Delta p}{\mu_{ef2}(L-\xi)} \tag{29}$$

and note that

$$F(0)=\frac{k_2\Delta p}{\mu_{ef2}(L-\xi)}<0, \qquad F\left(\frac{k_2\Delta p}{\mu_{ef2}M\xi}\right)=\frac{k_2\Delta p}{\mu_{ef}M\xi}>0. \tag{30}$$

Since a starting positive value for $\xi$ is provided by relation (28), we now are able to solve equations (23) and (24) simultaneously by combining two numerical methods: one for the transcendental equation (23) and one for the differential equation (24).

To clarify our numerical procedure, let us suppose that we have determined the interface position $\xi(t)$ for certain values of time, $t_0, t_1, \ldots, t_n$. For example, for $t_0=0$, $\xi(0)=0$ and, if $t_1$ is a short time for which (28) provides a reasonable approximation $\xi(t_i)$, we have at the starting stage $\xi(t_0)$ and $\xi(t_i)$. For each $\xi(t_i)$ evaluated previously, equation (23) may be solved numerically using the fact that $F(Z)$ changes sign in the interval

$$\left(0, \frac{k_1}{\mu_{ef1}}\frac{\Delta p}{\xi(t_i)}\right),$$

so that the corresponding $Z_i$ may be determined. Further, applying the implicit multistep Adams–Moulton method[11] to equation (24), we can evaluate $\xi(t)$, for $t=t_n+\Delta t$. The numerical procedure described above has been applied for some cases of practical interest and the results are illustrated in Figures 2–5 (see Section 6).

## 4. PLANE RADIAL FLOW

For plane radial flow the following equations are valid:

$$-\frac{\partial p_1}{\partial R}=\frac{\mu_{ef1}}{k_1}v_1^{n_1}, \qquad -\frac{\partial p_2}{\partial R}=\frac{\mu_{ef2}}{k_2}v_2^{n_2} \tag{31}$$

and

$$\frac{1}{R}\frac{\partial}{\partial R}(Rv_i)=0, \quad i=1, 2. \tag{32}$$

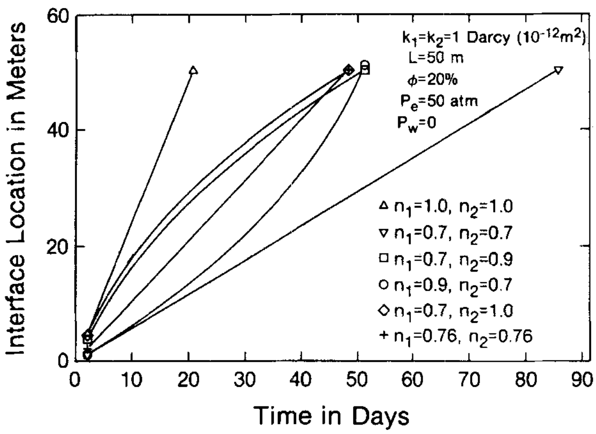


Figure 2. Effect of power law exponent on interface location for linear displacement mechanism

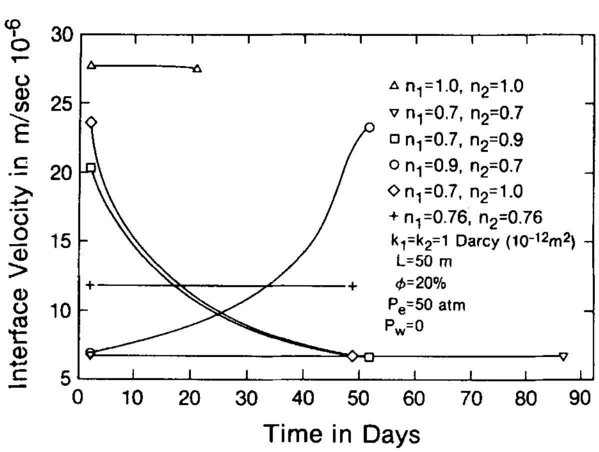


Figure 3. Effect of power law exponent on interface velocity for linear displacement mechanism

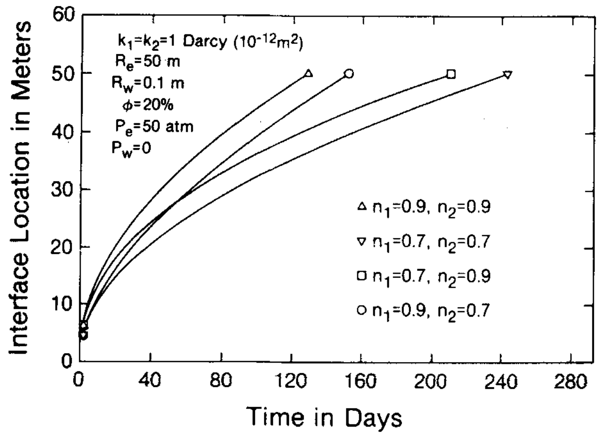


Figure 4. Effect of power law exponent on interface location for radial displacement mechanism

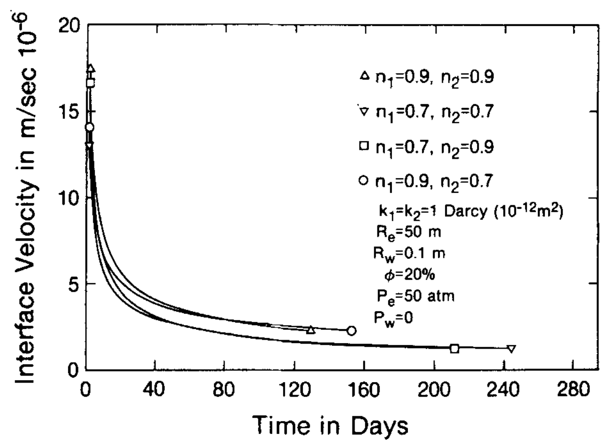


Figure 5. Effect of power law exponent on interface velocity for radial displacement mechanism.

From (31) and (32) it turns out that

$$\frac{\partial^2 p_1}{\partial R^2} + \frac{n_1}{R}\frac{\partial p_1}{\partial R} = 0, \quad R_w < R < \xi(t), \tag{33}$$

$$\frac{\partial^2 p_2}{\partial R^2} + \frac{n_2}{R}\frac{\partial p_2}{\partial R} = 0, \quad \xi(t) < R < R_e. \tag{34}$$

It should be noted that the only flow system giving plane radial flow is that bounded by two cylinders on which the pressure is uniform. Obviously this case includes the situation of a centrally located well in an oil reservoir where fluid is injected at a constant pressure. As a result, the following boundary conditions arise naturally:

$$\text{at} \quad R=R_w, \qquad p_1(R_w)=p_e,$$
$$\text{at} \quad R=R_e, \qquad p_2(R_e)=p_w, \quad p_e>p_w, \tag{35}$$

while at the moving interface $R=\xi(t)$ we have the conditions specified in (3) and (4).

Equations (33)–(35), (3) and (4) determine the pressure distributions expressed as

$$p_1(R)=p_e-(p_e-p^*)\frac{R^{1-n_1}-R_w^{1-n_1}}{\xi^{1-n_1}-R_w^{1-n_1}}, \quad R_w<R<\xi(t), \tag{36}$$

$$p_2(R)=p^*-(p^*-p_w)\frac{R^{1-n_2}-\xi^{1-n_2}}{R_e^{1-n_2}-\xi^{1-n_2}}, \quad \xi(t)<R<R_e, \tag{37}$$

and consequently from (31)

$$v_1=\left(\frac{k_1(1-n_1)(p_e-p^*)}{\mu_{ef1}R^{n_1}(\xi^{1-n_1}-R_w^{1-n_1})}\right)^{1/n_1} \tag{38}$$

and

$$v_2=\left(\frac{k_2(1-n_2)(p^*-p_w)}{\mu_{ef2}R^{n_2}(R^{1-n_2}-\xi^{1-n_2})}\right)^{1/n_2}. \tag{39}$$

The velocity equality at the interface, i.e. $v_1=v_2$, leads to

$$\left(\frac{k_1(1-n_1)(p_e-p^*)}{\mu_1\xi^{n_1}(\xi^{1-n_1}-R_w^{1-n_1})}\right)^{1/n_1}=\left(\frac{k_2(1-n_2)(p^*-p_w)}{\mu_2\xi^{n_2}(R_e^{1-n_2}-\xi^{1-n_2})}\right)^{1/n_2} \tag{40}$$

Since the equations describing the mathematical model of plane radial flow are non-linear, a numerical procedure should be used.

Before solving the general case numerically, we will present, as for the one-dimensional flow, an analytical solution for the particular case $n_1=n_2=n$. In this case (40) yields

$$p^*(\xi)=\frac{(R_e^{1-n}-\xi^{1-n})p_e+(\xi^{1-n}-R_w^{1-n})p_w M}{R_e^{1-n}-\xi^{1-n}+M(\xi^{1-n}-R_w^{1-n})}, \tag{41}$$

where $M$ is given by relation (16). Introducing the function

$$\psi(\xi)=\left[1-M\left(\frac{R_w}{R_e}\right)^{1-n}+(M-1)\left(\frac{\xi}{R_e}\right)^{1-n}\right]^{1/n}, \tag{42}$$

the interface velocity $V$ determined by (38) is expressed as

$$V=\frac{1}{\xi}\left(\frac{(1-n)k_1 M\Delta p}{R_e^{1-n}\mu_{ef1}\psi(\xi)}\right)^{1/n}. \tag{43}$$

Relations (13) and (43) lead to the differential equation

$$\varphi\xi\frac{d\xi}{dt}=\left(\frac{(1-n)k_1 M\Delta p}{R_e^{1-n}\mu_{ef1}\psi(\xi)}\right)^{1/n}, \tag{44}$$

with $\xi(0)=R_w$ at $t=0$. $R_w$ being the well radius. Finally equation (44) yields

$$T(\xi)=\int_{R_w}^{\xi}\xi\psi(\xi)\,d\xi, \tag{45}$$

where the dimensionless $T$ is defined as

$$T=\left(\frac{k_1(1-n)M\Delta p}{\mu_{ef1}R_e^{1-n}}\right)^{1/n}\frac{t}{\varphi}. \tag{46}$$

The integral in (45) can be analytically performed if and only if at least one of the expressions $1/n$, $1/(1-n)$, $1/n(1-n)$ is an integer. Otherwise a numerical quadrature formula is required.

To validate our results, let us compare the case $n_1=n_2=1$ with that when $n_1=n_2=n$ and $n\to 1$. For $n_1=n_2=1$, i.e. the case when both fluids are Newtonian, the pressure distributions are expressed as

$$p_1=p_e+(p_e-p^*)\frac{\ln(R/\xi)}{\ln(R_w/\xi)},\quad R_w<R<\xi, \tag{47}$$

$$p_2=p^*-(p^*-p_w)\frac{\ln(R/\xi)}{\ln(R_e/\xi)},\quad \xi<R<R_e. \tag{48}$$

From the velocity equality at the interface, i.e. at $R=\xi$, one obtains

$$p^*(\xi)=\frac{p_eM\ln(\xi/R_w)+p_e\ln(R_e/\xi)}{\ln(R_e/\xi)+M\ln(\xi/R_w)}. \tag{49}$$

On the other hand, relation (41) corresponding to $n_1=n_2=n$ may be rewritten as

$$p^*(\xi)=\frac{p_e+(Mp_w-p_e)(\xi/R_e)^{1-n}-Mp_w(R_w/R_e)^{1-n}}{1+(M-1)(\xi/R_e)^{1-n}-M(R_w/R_e)^{1-n}}. \tag{50}$$

Since $R_w/R_e\ll 1$, $\xi/R_e<1$ and $n<1$, the following asymptotic expressions are valid:

$$\left(\frac{R_w}{R_e}\right)^{1-n}\simeq 1+(1-n)\ln\frac{R_w}{R_e}, \tag{51}$$

$$\left(\frac{\xi}{R_e}\right)^{1-n}\simeq 1+(1-n)\ln\frac{\xi}{R_e}. \tag{52}$$

Introducing (51) and (52) into (50), one obtains a relation identical to (49).

## 5. NUMERICAL SOLUTION

In this section we are concerned with the numerical solution of the coupled non-linear equations describing the dynamics of the moving interface in a radial flow geometry. For this purpose it is convenient to introduce the notation

$$Z=(1-n_1)\frac{k_1}{\mu_{ef1}}\frac{p_e-p^*}{\xi^{n_1}(\xi^{1-n_1}-R_w^{1-n_1})}, \tag{53}$$

so that (40) may be written in the form

$$F(Z)=Z^m+M\frac{1-n_2}{1-n_1}\frac{\xi^{n_1}(\xi^{1-n_1}-R_w^{1-n_1})}{\xi^{n_2}(R_e^{1-n_2}-\xi^{1-n_2})}Z-\frac{k_2(1-n_2)\Delta p}{\mu_{ef2}\xi^{n_2}(R_e^{1-n_2}-\xi^{1-n_2})}=0, \tag{54}$$

where $m=n_2/n_1$.

For the interface velocity we have the differential equation

$$\frac{d\xi}{dt}=\frac{1}{\varphi}Z^{1/n_1}, \tag{55}$$

with the initial condition $\xi(0)=R_w$. From (53) one has

$$p^*=p_e-\frac{\mu_{ef1}}{k_1(1-n_1)}\xi^{n_1}(\xi^{1-n_1}-R_w^{1-n_1})Z. \tag{56}$$

The physical considerations for short time, previously shown for the one-dimensional case, are also valid for radial flow. As a result, from (56) we have $(\xi^{1-n_1}-R_w^{1-n})Z\to 0$ when $t\to 0$, whereas from (54) and (55) it turns out that

$$\int_{R_w}^{\xi}\xi\left[1-\left(\frac{\xi}{R_e}\right)^{1-n_2}\right]^{1/n_2}d\xi=\frac{t}{\varphi R_e^{(1-n_2)/n_2}}\left(\frac{(1-n_2)k_2\Delta p}{\mu_{ef2}}\right)^{1/n_2}. \tag{57}$$

Since $\xi/R_e\ll 1$ and $n_2<1$, the approximate relation

$$\left[1-\left(\frac{\xi}{R_e}\right)^{1-n_2}\right]^{1/n_2}\simeq 1-\frac{1}{n_2}\left(\frac{\xi}{R_e}\right)^{1-n_2} \tag{58}$$

may be used in (57), yielding the equation

$$\frac{1}{2}\xi^2-\frac{\xi^{3-n_2}}{n_2(3-n_2)R_e^{1-n_2}}-\frac{t}{\varphi R_e^{(1-n_2)/n_2}}\left(\frac{(1-n_2)k_2\Delta p}{\mu_{ef2}}\right)^{1/n_2}=0. \tag{59}$$

This equation defines $\xi(t)$ for short time. We will now show that (59) has a positive root. Denoting

$$\psi(\xi)=\xi^2\left[\frac{1}{2}-\frac{1}{n_2(3-n_2)}\left(\frac{\xi}{R_e}\right)^{1-n_2}\right]-\frac{1}{\varphi R_e^{(1-n_2)/n_2}}\left(\frac{(1-n_2)k_2\Delta p}{\mu_{ef2}}\right)^{1/n_2}t, \tag{60}$$

we have $\psi(0)<0$ for any $t>0$.

Taking $t=t_1$ small enough, one can find a $\xi^*$ for which $\psi(\xi^*)>0$ and therefore $\psi(\xi)=0$ has a root between 0 and $\xi^*$.

To find $\xi^*$, we note that if we choose for $\xi$ in (60) a value such that

$$\frac{1}{2}-\frac{1}{n_2(3-n_2)}\left(\frac{\xi}{R_e}\right)^{1-n_2}>\frac{1}{4}, \tag{61}$$

then

$$\psi(\xi)>\tfrac{1}{4}\xi^2-\frac{1}{\varphi R_e^{(1-n_2)/n_2}}\left(\frac{(1-n_2)k_2\Delta p}{\mu_{ef2}}\right)^{1/n_2}t_1. \tag{62}$$

On the other hand, taking in (62)

$$\xi^*=2\left(\frac{t_1}{\varphi R_e^{(1-n_2)/n_2}}\right)^{1/2}\left(\frac{(1-n_2)k_2\Delta p}{\mu_{ef2}}\right)^{1/2n_2}, \tag{63}$$

we have $\psi(\xi^*)>0$. $\zeta^*$ defined by (63) should also satisfy inequality (61), from which one finds a condition for $t_1$ which is expressed as

$$t_1\leqslant\tfrac{1}{4}\varphi R_e^{(1+n_2)/n_2}\left(\frac{n_2(3-n_2)}{4}\right)^{2/(1-n_2)}\left(\frac{\mu_{ef2}}{(1-n_2)k_2\Delta p}\right)^{1/n_2}. \tag{64}$$

Now one can solve equation (59) numerically knowing that $\psi(\xi)$ changes sign in the interval $(0, \xi^*)$. At this point we have found $\xi(0)$ and $\xi(t_1)$ for the coupled equations (54) and (55).

To determine the interface location $\xi(t)$ for further time steps, one can use, as for the one-dimensional flow, a combination of numerical methods in order to solve the transcendental equation (54) and the differential equation (55). For this purpose we note that in (54) $F(Z)$ changes sign in the interval

$$\left(0, \frac{(1-n_1)k_1\Delta p}{\mu_{\mathrm{ef}\,1}\xi^{n_1}(\xi^{1-n_1}-R_{\mathrm{w}}^{1-n_1})}\right),$$

so that the root of $F(Z)=0$ can be evaluated for each $\xi(t)$ known.[12] The implicit multistep Adams–Moulton method[11] applied to (55) will provide us with an evaluation of the interface position for a new time step.

## 6. DISCUSSION OF RESULTS

In order to illustrate the behaviour of the numerical procedure presented in Sections 3 and 5, the following example, taken from practice, has been considered: $L=50$ m, $\varphi=0{\cdot}2$, $k_1=k_2=1$ Darcy, $p_{\mathrm{e}}=50$ atm, $p_{\mathrm{w}}=0$, $R_{\mathrm{e}}=50$ m and $R_{\mathrm{w}}=10^{-1}$ m.

From various rheological data published in the literature, a relationship between $H$ and $n$ has been established by a fitting procedure and expressed as

$$H=(0{\cdot}1+0{\cdot}08/n^7)/100.$$

The case corresponding to one-dimensional flow is presented in Figures 2 and 3. Figure 2 shows the interface position as a function of time, while Figure 3 shows the interface velocity behaviour. The values of the power law exponents $n_1$ and $n_2$ are indicated in the figures.

The plane radial flow case is presented in Figures 4 and 5 for the same data as used in one-dimensional flow.

The particular case $n_1=n_2=n$, which was analytically solved in Sections 2 and 4, has also been shown in Figures 2–5 in order to validate the numerical approach shown in Sections 3 and 5 for $n_1\neq n_2$. Perfect agreement between the data obtained from analytical solutions and those obtained from numerical solutions was found.

Figures 2–5 illustrate the rheological effects of power law displacing and displaced fluids on the dynamics of a moving interface based on the advance frontal theory for non-Newtonian fluids.

A relevant result observed in the interface velocity behaviour (see Figure 3) is that for one-dimensional flow when $n_1>n_2$ the interface movement is accelerated, whereas when $n_1<n_2$ it is decelerated. The case $n_1=n_2$ leads to a uniform movement, i.e. at a constant interface velocity, as expected.

From the results shown in Figures 2–5 it is clear that the dynamics of the moving interface involved in oil displacement mechanics with non-Newtonian fluids of power law behaviour may be efficiently controlled by means of an appropriate strategy of optimal selection of rheological parameters of the injected fluid, expressed in terms of displaced fluid and reservoir properties.

## 7. CONCLUDING REMARKS

In this investigation we have presented a model based on the frontal advance theory for describing the dynamics of non-Newtonian fluid interfaces involved in oil displacement mechanics in a porous medium. A system of coupled non-linear equations governing the flow mechanism has been obtained and solved numerically. From the numerical solutions the interface location and its advance velocity as a function of time may be determined. The cases of linear and radial flow geometries under conditions of constant pressure of injection and production have been considered.

Several numerical examples of practical interest in oil reservoir engineering have been illustrated in Figures 2–5, from which it turns out that the interface movement may be accelerated or decelerated depending on the values of the power law exponents of the displacing and displaced fluids.

Based on the results obtained in this investigation, it is obvious that the dynamics of non-Newtonian fluid interfaces of power law behaviour may be controlled. For this purpose a strategy of optimal selection of the rheological parameters of the displacing fluid expressed in terms of displaced fluid and reservoir properties is required.

## APPENDIX I: NOMENCLATURE

| | |
|---|---|
| $H$ | consistency index, rheological parameter in power law model |
| $k$ | permeability |
| $L$ | length of the flow system |
| $M$ | mobility ratio |
| $n$ | power law exponent, rheological parameter in power law model |
| $p$ | pressure |
| $p_e$ | injection pressure |
| $p_w$ | production pressure |
| $R$ | radial distance |
| $R_e$ | external radius |
| $R_w$ | well radius |
| $t$ | time |
| $T$ | dimensionless time |
| $v$ | fluid velocity in porous medium |
| $V$ | interface velocity |

*Greek letters*

| | |
|---|---|
| $\varphi$ | porosity |
| $\xi(t)$ | interface position |
| $\mu$ | viscosity |
| $\mu_{ef}$ | effective viscosity |

*Subscript*

| | |
|---|---|
| 1 | refers to the displacing fluid |
| 2 | refers to the displaced fluid |

## APPENDIX II

To find a root of the equation

$$F(z)=0$$

in the interval $(a, b)$ knowing that $F(a)F(b)<0$, the Brent algorithm[12] can be used. This algorithm is a combination of linear interpolation, inverse quadratic interpolation and bisection.

A subroutine based on this algorithm with a superlinear convergence has been constructed and is available in the International Mathematical and Statistical Library (IMSL) under the name 'ZBRENT'.

The implicit multistep Adams–Moulton method for the equation

$$y' = f(y) \tag{65}$$

with the initial condition

$$\mathbf{y}(t_0) = y_0 \tag{66}$$

is given by the formula

$$y_n = \sum_{i=1}^{k} (\alpha_i y_{n-i} + \beta_i h y'_{n-i}) + \beta_0 h f(y_n), \tag{67}$$

where $y_j = y(t_j)$, $y'_j = f(t_j, y_j)$, $j = 0, 1, \ldots, n$, $1 < k < n$, assuming that $y_{n-i}$ and therefore $y'_{n-i}$, $i = 1, 2, \ldots, k$, have been previously calculated. The truncation error in (67) is $\gamma_k h^{k+1}$, i.e. of order $k+1$. The coefficients $\alpha_i$, $\beta_i$ and $\gamma_k$ can be found with the procedure given in Reference 11.

Equation (67) is an implicit equation in $y_n$, since $F(y_n)$ is non-linear in $y_n$. A predictor corrector process is employed to solve (67) as described in Reference 11. Based on the approach described above, a subroutine named 'DGEAR'[13] of order up to twelve has been constructed and is available in the International Mathematical and Statistical Library (IMSL).

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